Other conditions for the existence of a solution of system (1.3) can be mentioned. For instance, for $g \in C(\Omega)$ $(0 < \alpha \leqslant 1)$ and sufficiently small λ_i the Schauder principle can be used, as is done in /1/.

Different approximate methods /6, 8/ can be used to solve system (1.3). It should be taken into account here that the operator Q is Fréchet-differentiable only in certain sets $\omega \subset L_p$. For instance, (as an operator acting from $C(\Omega) \subset L_p(\Omega)$ into $L_p(\Omega)$, it is differentiable in the set

$$\omega = \{v : v \in C(\Omega),\ \mathrm{mes}\,\{M : v(M) = 0\} = 0\}$$

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INVERSE CONTACT PROBLEMS OF THE THEORY OF PLASTICITY*

V.I. KUZ'MENKO

A class of inverse contact problems of the theory of plasticity dealing with the determination of the form of a stamp ensuring the prescribed final change in the body shape is studied. The problem is given in the form of a functional equation. The principle of compressive mapping is used to show the existence and uniqueness of the solution, and an iterative process is given for determining the required form of the stamp. A problem dealing with the form of the stamp ensuring the formation of trapezoidal indentations in the strip surface is solved as an example.

1. Formulation of the problem. We shall connect a monotonically increasing parameter t, $t \in [0, T]$, with the process of quasistatic deformation of an elastoplastic body Ω, and we shall call it time. We use, as the spatial frame of reference, the Cartesian coordinate system $0x_1x_2x_3$. The symbols $u_i(x, t), \varepsilon_{ij}(x, t), \sigma_{ij}(x, t)$ denote the components of the vector of small displacements and of the small deformations and stress tensors at the point $x = (x_1, x_2, x_3)$, at the instant t.

The body Ω is bounded by a piecewise smooth surface composed of three parts: Γ_u, Γ_σ, Γ_c. The body is clamped over the surface Γ_u and the part Γ_σ is stress-free. The surface Γ_c is acted upon by the moving stamp. We describe the form of the stamp surface by the function $f(x)$ equal to the distance from the surface Γ_c to the stamp surface along the normal to Γ_c, at $t = 0$. The law of motion of the stamp as a rigid body is assumed given, does not depend on the form of the stamp, and must be chosen so that when $t < t^*$, an elastoplastic deformation takes place in the body Ω, while at $t \geqslant t^*$ we only have unloading or active elastic deformation. We assume that there is no contact whatsoever between the body and the stamp at $t = T$.

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Let us formulate the conditions of interaction between the body and the stamp during the deforming process. We can construct, uniquely, for the given function $f(x)$ determining the form of the stamp and for the given law of motion of the stamp as a rigid body, a function $\Phi(x, t)$ equal to the distance between the surface Γ_c and the stamp surface measured along the normal to Γ_c, at the instant t. We neglect the friction between the surface of the body and the stamp. The subscripts ν and τ will denote the normal and tangential components. Then the interaction between the body and the stamp will be characterized by the conditions /1, 2/

$$\sigma_\nu(x, t) \leqslant 0, \quad \sigma_\tau(x, t) = 0, \quad u_\nu(x, t) \leqslant \Phi(x, t) \tag{1.1}$$
$$\sigma_\nu(x, t)\,[u_\nu(x, t) - \Phi(x, t)] = 0, \quad \forall x \in \Gamma_c, \quad \forall t \in [0, T]$$

The relation between the stress and deformation states will be described by linear or non-linear differential relations written in terms of the increments thus:

$$d\sigma_{ij} = A_{ijkm}(\varkappa_1, \varkappa_2, \ldots, \varkappa_r, d\varepsilon_{\xi\eta})\, d\varepsilon_{km} \tag{1.2}$$

where A_{ijkm} are continuously differentiable functions of their arguments, homogeneous in zero degree in $d\varepsilon_{\xi\eta}$ and $\varkappa_1, \varkappa_2, \ldots, \varkappa_r$ are the values of certain functionals of the history of deformation. The conditions imposed on the functions A_{ijkm} were formulated in /2/.

We assume that the plastic deformation does not alter the elastic characteristics of the material, and we adopt the following linear law connecting the stresses and deformations under active elastic deformation and unloading:

$$d\sigma_{ij} = C_{ijkm}\, d\varepsilon_{km} \tag{1.3}$$

Also, $c > 0$ exists such that

$$C_{ijkm}\, d\varepsilon_{ij}\, d\varepsilon_{km} \geqslant c\, d\varepsilon_{ij}\, d\varepsilon_{ij} \tag{1.4}$$

Let the parameters $\varkappa_1, \varkappa_2, \ldots, \varkappa_l\ (l \leqslant r)$ characterize the hardening of the material. We assume that $\beta > 1$ can be found for any $\delta > 0$ such, that

$$A_{ijkm}(\varkappa_1, \varkappa_2, \ldots, \varkappa_r, \quad d\varepsilon_{\xi\eta})\, d\varepsilon_{ij}\, d\varepsilon_{km} \leqslant \beta C_{ijkm}\, d\varepsilon_{ij}\, d\varepsilon_{km} \tag{1.5}$$

for $\varkappa_1^2 + \varkappa_2^2 + \ldots + \varkappa_l^2 > \delta$, and the value of β does not depend on the history of the deformation. Condition (1.5) holds for most materials, and in the case of uniaxial deformation processes it means, in particular, that the tangential modulus must be smaller than the elastic modulus.

We will now give a direct formulation of the inverse contact problem (ICP). We shall specify the form of the surface Γ_c in the final state using the function $f_0(x), x \in \Gamma_c$ whose values will be equal to the distances between the initial, undeformed surface Γ_c, and the same surface in its final state. As earlier, the distance will be measured along the direction of the outer normal to the undeformed state of the surface Γ_c. Then the ICP of the theory of plasticity will be formulated as follows: to determine the function $f(x)$ so that the form of the surface Γ_c in its final state after plastic deformation and unloading is described by the function $f_0(x)$.

2. Formulation of the functional equation. Let us first introduce some mathematical concepts for subsequent use. We shall regard $[H^1(\Omega)]^3$ as a Sobolev space of vector functions $v(x) = (v_1(x), v_2(x), v_3(x))$ defined in Ω and square summable together with its first partial derivatives. We shall also introduce the spaces $H^{1/2}(\Gamma_c)$, $H^{-1/2}(\Gamma_c)$ of functions defined on the surface Γ_c with the corresponding norm /3/. We shall regard the elements of $H^{1/2}(\Gamma_c)$ as normal displacements of the points of the surface Γ_c, and the elements of $H^{-1/2}(\Gamma_c)$ as normal stresses on Γ_c.

Let us now consider the auxilliary direct problem which will be used in the formulation, study and solution of the functional equation of ICP.

Problem 1. We will use the relations of the theory of plasticity (1.2). The zero boundary conditions are given in terms of displacements and stresses on the surfaces Γ_u and Γ_σ respectively, and the surface Γ_c is acted upon by the moving stamp. The form of the stamp must be chosen in such a manner, that when the stamp moves according to the given law, the distance between the surface Γ_c and the stamp surface at the time t^* is equal to the value of the given function $\psi(x)$. If there is no contact at the points $x \in \Gamma_c$ at the time $t = t^*$, then the points undergo normal displacements $\psi(x) - u_\nu(x, t^*)$ according to a definite law. We require to determine the stress-deformation state in the body Ω and the distribution of the normal contact stresses on Γ_c at the instant t^*.

Problem 2 is formulated just like problem 1, but the stresses and deformations are connected by the relations of the theory of elasticity (1.3).

Problem 3 represents the following problem of the linear theory of elasticity. The following forces are given on Γ_c:

$$\sigma_\nu(x) = q(x), \quad \sigma_\tau(x) = 0$$

and the zero displacements and stresses on Γ_u and Γ_σ, respectively. We require to determine the stress-deformation state in the body Ω and normal displacements of the points of the surface Γ_c.

Every one of the above problems has a unique solution /1, 2/, and the normal displacements and normal contact stress distributions obtained belong to the spaces $H^{1/2}(\Gamma_c)$ and $H^{-1/2}(\Gamma_c)$, respectively.

Let $u^{(1)} \in [H^1(\Omega)]^3$ be a solution of Problem 2. We define the components of the deformation tensor by the Cauchy relations and introduce the function of the deformation energy density

$$W(\varepsilon_{ij}^{(1)}) = {}^1/_2 C_{ijkm} \varepsilon_{ij}^{(1)} \varepsilon_{km}^{(1)} \tag{2.1}$$

Then we can place, in a unique manner, the number

$$\|\psi\| = \left(\int_\Omega W(\varepsilon_{ij}^{(1)})\, d\Omega\right)^{1/2} \tag{2.2}$$

in 1:1 correspondence with every function $\psi \in H^{1/2}(\Gamma_c)$.

Since all axioms of the norm hold for $\|\psi\|$, it follows that the elements $\psi \in H^{1/2}(\Gamma_c)$ can be regarded as the elements of the Banach space U with the norm (2.2).

Similarly, having solved Problem 3 and having determined $\varepsilon_{ij}^{(2)}$, we introduce the norm of the elements $q \in H^{-1/2}(\Gamma_c)$ as follows:

$$\|q\| = \left(\int_\Omega W(\varepsilon_{ij}^{(2)})\, d\Omega\right)^{1/2} \tag{2.3}$$

and denote the corresponding Banach space of normal stresses with norm (2.3), by S.

Let us introduce certain operators acting in the spaces U and S. The operator Q_p: $U \to S$ places in 1:1 correspondence with every function $\psi \in U$ the distribution of normal stresses $q \in S$, obtained by solving Problem 1. The action of the operator Q_e: $U \to S$ differs from that of Q_p in the fact that the distribution of normal stresses $q \in S$ is obtained while solving Problem 2. Finally, the operator Q_e^{-1}: $S \to U$ is the inverse of Q_e and places in 1:1 correspondence with the functions $q \in S$ the displacements $\psi \in U$ obtained by solving Problem 3.

Using the above operators we will formulate the problem in the form of a functional equation.

The normal contact stresses at the instant t^* at which the unloading starts, are determined using the operators given above as follows: $q^*(x) = Q_p[\psi(x)]$. Let us now subject the points of the surface Γ_c in its final state, to normal displacements $\psi(x) - f_0(x)$. Assuming that the accompanying deformations are elastic, we obtain the corresponding normal stresses $q^{**}(x) = Q_e[\psi(x) - f_0(x)]$.

Assuming now that the elastic deformation is reversible, we use the theorem on unloading under contact interaction /4/ to conclude that the relation $q^*(x) = q^{**}(x)$, $\forall x \in \Gamma_c$ must hold for the function $\psi(x)$ corresponding to the required form of the stamp $f(x)$, or

$$Q_p[\psi(x)] = Q_e[\psi(x) - f_0(x)] \tag{2.4}$$

Applying the operator Q_e^{-1}, we write Eq.(2.4) in the form

$$\psi(x) = P[\psi(x)], \quad P[\psi] = f_0 + Q_e^{-1} Q_p \psi \tag{2.5}$$

Thus the solution of ICP of the theory of plasticity is reduced to the solution of the functional Eq.(2.5) for the normal displacements $\psi(x)$ of the points of the surface Γ_c at the instant at which the unloading starts.

3. Existence and uniqueness of the solution of ICP. The study of the correctness of the formulation of the problem and the construction of the method of solution are based on the principle of compressive mapping /5/.

Let us separate, in the space U, a closed convex set U_0 of elements $\psi \in U$ satisfying the inequality

$$\|\psi - f_0 - Q_e^{-1} Q_p \psi\| \leqslant a \|f_0\|, \quad a < 1 \tag{3.1}$$

It is evident that the solution of the functional equation sought belongs to the set U_0.

Let us denote by $\Omega_p(\psi) \subset \Omega$ the domain of active plastic deformations, at all the points of which $\varkappa_1^2 + \varkappa_2^2 + \ldots + \varkappa_l^2 > \delta$, and hence where the inequality (1.5) holds.

Lemma 1. A $\gamma_0 > 0$ exists such that the following inequality holds for all $\psi \in U_0$:

$$\operatorname{mes}(\Omega_p(\psi)) \geqslant \gamma_0 > 0$$

We prove it by assuming the opposite. Assume that a function $\psi_n \in U_0$ can be found for any $\gamma_n > 0$ such that $\operatorname{mes}(\Omega_p(\psi)) < \gamma_n$. Selecting a sequence $\{\gamma_n\}$ converging to zero we find that $\lim_{n\to\infty} \operatorname{mes}(\Omega_p(\psi_n)) = 0$. But, if only an elastic deformation occurs at almost all points of the body, then $Q_e^{-1} Q_p \psi = \psi$ and hence

$$\|\psi - f_0 - Q_e^{-1}Q_p\psi\| = \|f_0\|$$

i.e. N can be found such that the functions ψ_n will not satisfy condition (3.1) for all $n > N$.
Let us denote by $P'(\psi)$ the Fréchet derivative of the operator P at the point $\psi \in U_0$.

Lemma 2. When the assumptions made earlier hold, the following inequality is satisfied:

$$\sup_{\psi \in U_0} \|P'(\psi)\| \leqslant \alpha < 1$$

Proof. We denote by $d\varepsilon_{ij}$, $d\sigma_{ij}$ the deformation and stress increments obtained as a result of solving Problem 2 of the theory of elasticity, when the increments in normal displacements $d\psi$ are specified on Γ_c. Then

$$\|d\psi\| = \left(\frac{1}{2}\int_\Omega C_{ijkm} d\varepsilon_{ij} d\varepsilon_{km} d\Omega\right)^{1/2} = \left(\frac{1}{2}\int_\Omega d\sigma_{ij} d\varepsilon_{ij} d\Omega\right)^{1/2}$$

Let $D\psi$ be the Fréchet differential of the operator P at the point $\psi \in U_0$. Subsequent proof consists of obtaining an estimate of the form $\|D\psi\| \leqslant \alpha \|d\psi\|$, $\alpha < 1$, and includes four consecutive stages.

1°. Using the components of the tensor $d\varepsilon_{ij}$ of deformation increments obtained in the course of solving Problem 2, we define formally, at the points of the region $\Omega_p(\psi)$, the components of some tensor $d\sigma_{ij}^{(1)}$ of stress increments with help of relations (1.2) of the theory of plasticity. We write $d\sigma_{ij}^{(1)} = d\sigma_{ij}$ in the region $\Omega - \Omega_p(\psi)$. From assumption (1.5) it follows that in Ω_p, $d\sigma_{ij}^{(1)} d\varepsilon_{ij} \leqslant \beta_1 d\sigma_{ij} d\varepsilon_{ij}$ almost everywhere. Then

$$\begin{gathered}\frac{1}{2}\int_\Omega d\sigma_{ij}^{(1)} d\varepsilon_{ij} d\Omega = \frac{1}{2}\int_{\Omega_p(\psi)} d\sigma_{ij}^{(1)} d\varepsilon_{ij} d\Omega + \frac{1}{2}\int_{\Omega-\Omega_p(\psi)} d\sigma_{ij} d\varepsilon_{ij} d\Omega \leqslant \\ \frac{1}{2}\beta_1 \int_{\Omega_p(\psi)} d\sigma_{ij} d\varepsilon_{ij} d\Omega + \frac{1}{2}\int_{\Omega-\Omega_p(\psi)} d\sigma_{ij} d\varepsilon_{ij} d\Omega = \alpha_1 \|d\psi\|^2 \\ \alpha_1 = 1 - (1-\beta_1)\int_{\Omega_p(\psi)} d\sigma_{ij} d\varepsilon_{ij} d\Omega \Big/ \int_\Omega d\sigma_{ij} d\varepsilon_{ij} d\Omega\end{gathered} \tag{3.2}$$

Using Lemma 1 we conclude that $\alpha_1 < 1$.

2°. Let us denote by $d\varepsilon_{ij}^{(2)}$, $d\sigma_{ij}^{(2)}$ the solution of the problem of the theory of plasticity corresponding to the increments in normal displacements $d\psi$ on Γ_c specified at $t = t^*$. According to the extremal principle of displacement increments /6/, when no forces act on Γ_σ, the functional

$$\frac{1}{2}\int_\Omega d\sigma_{ij}^* d\varepsilon_{ij} d\Omega$$

reaches its minimum value for the real increments $d\varepsilon_{ij}^{(2)}$, $d\sigma_{ij}^{(2)}$ compared with all possible corresponding given increments $d\psi$ on Γ_c. We can use, in particular, $d\varepsilon_{ij}$, $d\sigma_{ij}^{(1)}$, as such possible increments. We therefore have

$$\frac{1}{2}\int_\Omega d\sigma_{ij}^{(2)} d\varepsilon_{ij}^{(2)} d\Omega \leqslant \frac{1}{2}\int_\Omega d\sigma_{ij}^{(1)} d\varepsilon_{ij} d\Omega \leqslant \alpha_1 \|d\psi\|^2 \tag{3.3}$$

3°. We write $d\varepsilon_{ij}^{(3)} = C_{ijkm}^{-1} d\sigma_{km}^{(2)}$. Clearly, we have in the region of elastic deformations $d\varepsilon_{ij}^{(3)} = d\varepsilon_{ij}^{(2)}$, and in the region $\Omega_p(\psi)$ inequality (1.5) takes the form

$$d\sigma_{ij}^{(2)} d\varepsilon_{ij}^{(3)} \leqslant \beta_2 d\sigma_{ij}^{(2)} d\varepsilon_{ij}^{(2)}, \quad \beta_2 < 1$$

We arrive, as in stage 1°, at the following inequality:

$$\begin{gathered}\frac{1}{2}\int_\Omega d\sigma_{ij}^{(2)} d\varepsilon_{ij}^{(3)} d\Omega \leqslant \frac{1}{2}\alpha_2 \int_\Omega d\sigma_{ij}^{(2)} d\varepsilon_{ij}^{(2)} d\Omega \\ \alpha_2 = 1 - (1-\beta_2)\int_{\Omega_p(\psi)} d\sigma_{ij}^{(2)} d\varepsilon_{ij}^{(2)} d\Omega \Big/ \int_\Omega d\sigma_{ij}^{(2)} d\varepsilon_{ij}^{(2)} d\Omega < 1\end{gathered} \tag{3.4}$$

Combining (3.3) and (3.4), we obtain the estimate

$$\frac{1}{2}\int_\Omega d\sigma_{ij}^{(2)} d\varepsilon_{ij}^{(3)} d\Omega \leqslant \alpha_1\alpha_2 \|d\psi\|^2 \tag{3.5}$$

4°. Let $dq^{(2)}$ be the normal stresses at the boundary obtained as a result of solving the problem of the theory of plasticity, when the normal displacement increments $d\psi$ are given on Γ_c. Assuming that the stresses $dq^{(2)}$ are known, we shall consider the problem of the theory of elasticity, with the normal stresses given on Γ_c. Let us denote by $d\varepsilon_{ij}^{(4)}, d\sigma_{ij}^{(4)}$ the deformation and stress increments obtained in the course of solving the problem. Using the extremal principle principle of stress increments, we obtain the inequality

$$\frac{1}{2}\int\limits_{\Omega} d\sigma_{ij}^{(4)}\, d\varepsilon_{ij}^{(4)}\, d\Omega \leqslant \frac{1}{2}\int\limits_{\Omega} d\sigma_{ij}^{(2)}\, d\varepsilon_{ij}^{(3)}\, d\Omega \leqslant \alpha_1\alpha_2 \| d\psi \|^2 \tag{3.6}$$

We will now turn our attention to the fact that the increments $d\varepsilon_{ij}^{(4)}, d\sigma_{ij}^{(4)}$ are obtained from $d\varepsilon_{ij}^{(2)}$, $d\sigma_{ij}^{(2)}$ as a result of the action of the operator Q_e^{-1}, and $d\varepsilon_{ij}^{(2)}, d\sigma_{ij}^{(2)}$ are obtained in turn by the action of the operator Q_p. Thus the increments $d\varepsilon_{ij}^{(4)}$, $d\sigma_{ij}^{(4)}$ correspond to the increments $d\psi^{(4)}$ resulting from the action of the operator $Q_e^{-1}Q_p$. Assuming now that $Df_0 = 0$, we can regard the quantity

$$\left(\frac{1}{2}\int\limits_{\Omega} d\sigma_{ij}^{(4)}\, d\varepsilon_{ij}^{(4)}\, d\Omega\right)^{1/2}$$

as the norm $\|D\psi\|$. Using the estimate (3.6) we conclude that

$$\|D\psi\| \leqslant \alpha \|d\psi\|, \quad \alpha = \sqrt{\alpha_1\alpha_2}, \quad \forall\psi \in U_0$$

and hence

$$\sup_{\psi\in U_0} \|P'(\psi)\| = \sup_{\psi\in U_0} \frac{\|D\psi\|}{\|d\psi\|} \leqslant \alpha < 1$$

Theorem 1. Under the assumptions made, there exists a unique solution $\psi_* \in U_0$ of the functional Eq.(2.5), and ψ_* can be obtained as the limit of the sequence $\{\psi_n\}$, constructed with help of the recurrence relation

$$\psi_{n+1} = P(\psi_n) \quad (n = 0, 1, 2, \ldots) \tag{3.7}$$

where ψ_0 denotes any elements of U_0.

Proof. We shall utilize the formula of finite increments /5/ in the form

$$\| P(\psi_1) - P(\psi_2) \| \leqslant \| \psi_1 - \psi_2 \| \sup_{0<\theta<1} \| P'(\psi_2 + \theta(\psi_1 - \psi_2)) \|$$

and using Lemma 2 we conclude that

$$\| P(\psi_1) - P(\psi_2) \| \leqslant \alpha \| \psi_1 - \psi_2 \| \tag{3.8}$$

The space U can be regarded as metric with the length $\rho(\psi_1, \psi_2) = \| \psi_1 - \psi_2 \|$. Then the inequality (3.8) will imply that the operator P is a compression operator /5/. The statement of the theorem follows at once from the principle of compressive mappings /5/.

Theorem 1 refers to the problem of solving the functional Eq.(2.5). Using this theorem we will consider the problems of the existence and uniqueness of the solution of ICP. Let ψ_* be a solution of the functional Eq.(2.5). With the law of motion of the stamp specified, we can place the function f_* describing the form of the stamp, in 1:1 correspondence with the function ψ_* in a unique manner. However, $f_*(x)$ will be the solution of ICP only when the normal contact stresses, in accordance with conditions (1.1), are non-positive at $t = t^*$ at almost all points of Γ_c. Thus we arrive at the following alternative assertion.

Theorem 2. In order to study the problem of the existence and uniqueness of the solution of ICP of the theory of plasticity, and to actually find the required form of the stamp, we must solve the functional Eq.(2.5). If, for the solution $\psi_*(x)$ obtained the corresponding values of $q_*(x)$ are strictly positive on the set with non-zero measure in Γ_c, then the ICP has no solution. If $q_*(x) \leqslant 0$ for almost all $x \in \Gamma_c$, then the ICP of the theory of plasticity has a unique solution.

Corollary. The ICP of the theory of plasticity with the final form $f_{0\sigma}(x)$ of the free surface Γ_σ additionally specified, has no solutions for almost all given functions $f_{0\sigma}$.

Indeed, the ICP has a unique solution when only the final form of the surface Γ_c is given. In accordance with this solution, the final form of the free surface f_σ will also be unique and will not, in general, be the same as the given function $f_{0\sigma}$.

4. Numerical solution of the ICP. The method of solving the functional equation is in fact already given in Theorem 1, therefore we shall only consider certain particular aspects of its application and give the estimate for the rate of convergence.

In accordance with Theorem 1, the solution of functional Eq.(2.5) can be obtained by carrying out the iterative process

$$\psi_{n+1} = P(\psi_n) \quad (n = 0, 1, 2, \ldots), \quad \psi_0 \in U_0$$

Problems 1 and 3 formulated in Sect.2 are used in the realization of the operators Q_p

and Q_e^{-1} at every step of the iterative process. Existing numerical methods can be used effectively to solve such direct problems, e.g. the finite elements method and the boundary elements method.

The rate of convergence of the sequence $\{\psi_n\}$ obtained using relation (3.7) is characterized by the inequality /5/

$$\|\psi_n-\psi_*\|\leqslant\frac{\alpha^n}{1-\alpha}\|\psi_1-\psi_0\|$$

and is in fact given by the quantity $\alpha=\sqrt{\alpha_1\alpha_2}$. Let us consider the factors influencing α, using the estimates (3.2) and (3.4) obtained in the proof of Lemma 2. The constants β_1 and β_2 are proportional, in the case of uniaxial deformation, to the ratio of the tangential and elastic moduli. From (3.2) and (3.4) it follows that the rate of convergence increases as the

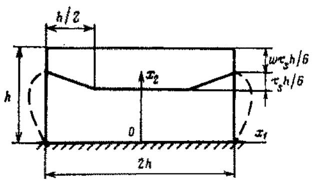

Fig.1

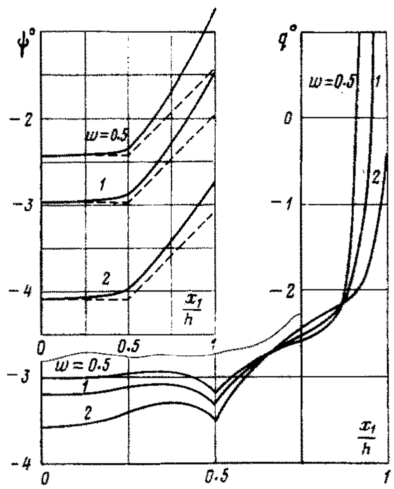

Fig.2

value of this ratio decreases. Further, the ratios of the integrals appearing in the estimates for α_1 and α_2 show that when the relative volume of the regions of active plastic deformations increases, so does the rate of convergence of the sequence $\{\psi_n\}$. We find that in the case of "deeper" impressions the rate of convergence is higher, since the region of plastic deformations increases, as a rule, with increasing depth of the impression.

The above method was used to write a set of programs for solving the ICP for an elasto-plastic strip of finite size. The direct Problems 1 and 3 were solved using the finite elements method based on the variational approach.

We shall consider as an example the problem of determining the form of the stamp for which a final trapezoidal indentation is formed (Fig.1) on the surface $x_2=h$ under the conditions of plane deformation. From the corollary of Theorem 2 it follows that the final form of the free surfaces cannot be specified in advance, and is shown in Fig.1 by the dashed line. The limiting torsional elasticity is denoted by τ_s and the shear modulus by G. The dimensionless parameter w characterizes the depth of the final impression. The stamp can only move translationally in the direction of the $0x_2$ axis.

The material of the strip is assumed to be homogeneous and isotropic. We use here the theory of small elastoplastic deformations for a linearly hardening material with the ratio of tangential to elastic modulus equal to 0.05. Poisson's ratio is taken to be equal to 0.3.

In the course of discretization of the direct problems the cross-section of the strip was divided into 800 finite rectangular elements. We require 4-7 iterations to determine the form of the stamp with a mean-square error not exceeding $0.001\tau_s h/G$, for $w=0.5$; 1; 1.5; 2; 2.5, and the number of iterations required decreases as w increases, which is in full agreement with the analysis of the rate of convergence carried out above.

Fig.2 shows the normal displacements $\psi_*(x_1)$ of the points on the surface Γ_c obtained in the course of solving the functional Eq.(2.5), and the corresponding normal stresses $q_*(x_1)$ on Γ_c for various values of w ($\psi^0=\psi_*G/(\tau_s h)$, $q^0=q_*/\tau_s$). The dashed lines show the form of the final imprint, displaced for clarity along the $0x_2$ axis. Note that the displacements obtained differ essentially from the form of the imprint. When $x_1\leqslant 0.5h$, the difference increases as w increases, and decreases when $x_1>0.5h$.

When $w<1.5$, the normal stresses are positive on the part $x_2=h$ of the surface. This leads us to the conclusion that from Theorem 2 it follows that an imprint of a given form can be obtained by impressing a stamp only when $w \geqslant 1.5$. In order to impress a stamp of given form to the required depth we must, for $w = 1.5; 2; 2.5$ apply the forces per unit width of the strip, equal to $5.58\tau_s h$, $5.9\tau_s h$, $6.22\tau_s h$.

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GENERALIZED SOLUTIONS IN THE THEORY OF PLASTICITY*

B.A. DRUYANOV

Conditions prevailing on the surfaces of the strong velocity discontinuities in rigid-plastic media were studied by many workers, e.g. /1, 2/. However, in all cases known to the author the conditions were obtained by utilizing a passage to the limit, when the surface of the discontinuity was considered as a limit to which a layer tends, the layer undergoing an intense deformation and its thickness tending to zero. Meanwhile, it is desirable to obtain the conditions at the discontinuities by intrinsic means from the system of equations itself, without bringing in the irrelevant concepts on what represents the surface of the discontinuity. To this end the equations must be given in divergent form. In the theory of plasticity the main difficulties in this respect are encountered in connection with the law of flow and the law controlling the hardening.

The present paper shows that certain generalization of the Mises principle makes it possible to impart to the inequality expressing it a divergent form and enables us to write it in integral form. From this it follows that in the incompressible plastic medium the surface of discontinuity in the tangential velocity component serves as the surface of maximum tangential stresses, with tangential stress directed along the velocity jump vector. In a compressible plastic medium the stress discontinuity is determined from the condition that the direction of the six-dimensional deformation velocity "vector" is continuous. We note that the integral form of the Mises inequality was used in /3/ to prove the existence and uniqueness of the solution. It was not, however, given in divergent form, and the conditions at the discontinuities were not considered.

With regard to the equation describing the hardening law, it can be reduced to divergent form when the specific plastic work is used as the hardening parameter.

The problem considered here is that of steady motion of a strip of finite thickness undergoing pure shear, in a rigid-plastic hardening medium. The emission of heat caused by plastic deformation and its effect

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